Analysis of Optimal Evasive Maneuvers Based on a Linearized Two-Dimensional Kinematic Model

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Optimal evasion from proportionally guided missiles is analyzed assuming two-dimensional linearized kinematics. By this assumption a simple search technique can be used instead of the cumbersome solution of a two-point boundary-value problem. Because of the simplicity of this approach, it is possible to include in the mathematical model factors which have been neglected in other analytic studies. It is demonstrated that these factors, e.g., the exact dynamic structure of the guidance system, the location of the saturating element in the guidance loop, the limited roll rate of the evading aircraft, etc., strongly affect the optimal maneuver sequence and determine the magnitude of the resulting miss distance. Comparison with other studies, which used two-dimensional nonlinear kinematic models, show that linearized kinematics and two-dimensional analysis have the same domain of validity. For optimal evasion assessment, the validity of both assumptions is limited to nearly "head-on" or "tail chase" situations. Engagements with other initial conditions require a three-dimensional modelling. The method described in this paper can be extended for such three-dimensional analysis.

I. Introduction

IN the last years several works have been addressed to the problem of optimal evasion from guided missiles. Unfortunately, the mathematical models used in these studies[1-6] were oversimplified and their results had a rather limited applicability. The motivation to use very simple models can be related to the following reasons: 1) the inherent complexity of the suggested control optimization techniques, and 2) the fact that guidance analysis, based on simple models, had provided useful missile design criteria.

A remarkable example for the second statement is the miss distance calculation of proportionally guided missiles. It was shown[7,8] that miss distances due to launching errors and constant target maneuvers practically vanish if two conditions are satisfied: 1) the effective navigation constant of the guidance N' is at least 3, and 2) the total smoothing time constant is less than 1/10 of the guided flight time.

This result, derived from the closed-form solution for a model of linearized two-dimensional kinematics and first-order dynamics, remains valid uniformly for stable guidance systems of any order. Moreover, these criteria actually have been used to determine the gain of the guidance loop and the minimum effective firing range of many operational missile systems.

Experience has shown that appropriate maneuvering can help to evade from guided missiles, designed according to those guidelines. Successful evasion depends on many factors which have been neglected in simplified models.[1-5]

Detailed simulation studies, proposed as an alternative, are both expensive and time consuming. They have to be used, probably, to validate evasive strategies, but by no means can be considered as a cost-effective way to develop one. The purpose of this paper is to analyze the major factors affecting the optimization of evasive maneuvers, and to generate realistic but still relatively simple models for future optimal evasion studies.

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Analysis is based on the following assumptions.

1) Missile and target are both considered as constant-speed mass points. The hypothesis of constant speed does not hurt the generality of the results, because in most cases longitudinal accelerations are not significant in the guided phase of the missile flight. The point-mass approximation has, however, some limitations. It is justified only if the resulting miss distances are either very small or very large. If the miss distance has a magnitude comparable to the target or missile dimensions, the exact aimpoint of the seeker and the activation logic of the proximity fuse also have to be considered in any hit assessment.

2) Missile is guided by proportional navigation, this being the guidance law implemented in the majority of operational weapon systems. However, the methods presented in this work can be applied without difficulty to other guidance systems as well.

3) Missile trajectory can be linearized around the initial collision course.

4) Engagement is confined to the horizontal plane (gravity is neglected).

These assumptions are strongly related to each other. Guiding a missile against an airborn target is an inherently three-dimensional process, described by nonlinear vector equations. However, these equations can be decomposed, as shown by Adler,[8] into two independent identical sets of equations, describing the guidance in two perpendicular planes, including the line of sight. This decomposition can be carried out only if a) the missile has two identical and decoupled channels of guidance, which implies roll-stabilized cruciform configurations, and b) the missile trajectory can be linearized around the initial collision course.

The second condition indicates that the assumption of two-dimensional analysis (no. 4) is a consequence of linearized kinematics (assumption no. 3).

5) Target lateral acceleration component, perpendicular to the initial line of sight, can be kept constant. This is another consequence of the trajectory linearization. As the line of sight turning rate is only affected by the normal acceleration component, this assumption provides an important simplification.

6) Target has complete information on the relative state (missile position and closing velocity) and on the missile parameters. This last assumption can be interpreted practically, that the pilot of the target airplane is alerted, that a

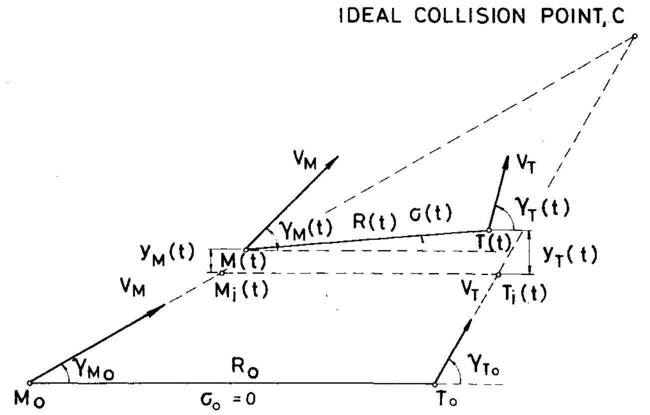

Fig. 1 Interception geometry.

missile of known type was launched against him, and he is able to estimate the relative position of the missile and the time to be intercepted. Although these conditions are not always satisfied, the assumption is necessary to define an optimal evasive strategy in a deterministic sense.

For this purpose, the problem of missile guidance against a maneuvering target is formulated as a fixed-time optimal control process, in which the objective is to maximize the miss distance. The mathematical model of linearized kinematics is used with three different missile dynamic structures: a) linear guidance dynamics (no acceleration limit), b) limited missile acceleration output (aerodynamic limit), and c) limited guidance command. Target dynamics are represented in first approximation by limited roll-rate.

The different dynamic models help to demonstrate the effect of various parameters on optimal evasion. Comparison with results of a nonlinear kinematic model[6] was carried out for validation.

Statement of the Problem – Modelling

A. Trajectory Linearization

The parameters of the missile/target interception geometry for zero launching error are defined in Fig. 1. The initial values γ_{T_0} and γ_{M_0} are related to each other by the collision course equation

$$V_T \sin(\gamma_{T_0}) = V_M \sin(\gamma_{M_0}) \tag{1}$$

To carry out trajectory linearization express

$$\gamma_T(t) = \gamma_{T_0} + \Delta\gamma_T(t) \tag{2}$$

$$\gamma_M(t) = \gamma_{M_0} + \Delta\gamma_M(t) \tag{3}$$

Assuming that $\Delta\gamma_T(t)$ and $\Delta\gamma_M(t)$ are small angles, leads to the following linearized differential equations:

$$\dot{R} \simeq V_T \cos(\gamma_{T_0}) - V_M \cos(\gamma_{M_0})$$

$$\triangleq V_{TR} - V_{MR} = -V_c = \text{const.} \tag{4}$$

$$\dot{y} \triangleq \dot{y}_T - \dot{y}_M = V_{TR}\Delta\gamma_T - V_{MR}\Delta\gamma_M \tag{5}$$

V_{TR} and V_{MR} being velocity projections on the initial line of sight.

Integrating Eq. (4) yields

$$R = R_0 - V_c t = V_c(t_f - t) \tag{6}$$

defining the predicted missile flight time t_f.

Assuming that the line of sight angle $\sigma(t)$ is small, the line of sight rotation can be expressed by

$$\dot{\sigma} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{R}\right) = \frac{1}{V_c(t_f - t)^2}[y + \dot{y}(t_f - t)] \tag{7}$$

The missile lateral acceleration perpendicular to the line of sight is commanded by the law of proportional navigation

$$(\ddot{y}_M)_c \triangleq V_{MR}(\dot{\gamma}_M)_c = N' V_c \dot{\sigma} \tag{8}$$

where N' is the effective navigation constant.

The dynamic relationship between the actual and the commanded missile maneuver is determined by the transfer function of the guidance loop (assumed to be linear and time invariant)

$$\frac{\ddot{y}_M}{(\ddot{y}_M)_c} = F(s) = \frac{1 + a_1 s + \ldots a_k s^k}{1 + b_1 s + \ldots b_p s^p} \quad (k < p) \tag{9}$$

Target acceleration perpendicular to the line of sight,

$$\ddot{y}_T \triangleq V_{TR}(\dot{\gamma}_T) \tag{10}$$

is bounded by the constraint

$$|\dot{\gamma}_T| \leq (\dot{\gamma}_T)_{\max} \tag{11}$$

In this linearized kinematic model the miss distance is defined by

$$m_0 = \lim_{R \to 0} (y_T - y_M) \triangleq y(t_f) \tag{12}$$

B. Formulation of the Optimal Evasion Problem

The state vector of the problem is defined as

$$x(t) = \text{col}[x_1, x_2, x_3, \ldots x_n]$$

$$\triangleq \text{col}[\Delta\gamma_T, y, \Delta\gamma_M, \dot{\gamma}_M] \tag{13}$$

and Eqs. (5-11) are presented in linear vector form

$$\dot{x}(t) = A(t)x(t) + bu(t) \tag{14}$$

with

$$b = (\dot{\gamma}_T)_{\max}\text{col}[1, 0, 0, \ldots 0] \tag{15}$$

The optimal control problem is of a fixed time t_f and its objective is to maximize a terminal payoff (square of the miss distance):

$$m_0^2 = x_2^2(t_f) \tag{16}$$

The control function to be optimized $u(t)$ is subject to the constraint (11)

$$|u(t)| \leq 1 \tag{17}$$

The Hamiltonian of this problem is

$$H(t,x,\lambda,u) = \lambda^T \cdot \dot{x} \triangleq H_0(t,x,\lambda) + (\dot{\gamma}_T)_{\max}\lambda_1 u(t) \tag{18}$$

H_0 being the part independent of the control function u.

The components of the costate vector are the solutions of the adjoint differential equation, which for a linear system is

$$\dot{\lambda}(t) = -A^T(t)\lambda(t) \tag{19}$$

with the boundary conditions

$$\lambda_i(t_f) = 0 \qquad (i \neq 2) \tag{20a}$$

$$\lambda_2(t_f) = 2x_2(t_f) = 2m_0 \tag{20b}$$

The optimal control function $u^*(t)$ is determined, as required by the maximum principle, by minimizing the Hamiltonian with respect to $u(t)$, resulting in

$$u^*(t) = -\operatorname{sign}[\lambda_l(t)] \tag{21}$$

As Eq. (19) is homogeneous and independent of x, the costate vector can be determined, up to a constant multiplier, without solving the state equation (14). The optimal control function is, therefore, independent of the trajectory. In the Appendix a closed-form solution of Eq. (19) is derived for a first-order system.

C. Limited Missile Maneuverability

The linear model presented in the previous subsection is based implicitly on the assumption of nonlimited missile maneuverability. Actually, every real missile system is subject to maneuverability saturation due to aerodynamic or structural constraints. The guidance system can be considered linear only if the required lateral acceleration does not attain the saturation level. It has been shown[15] that the solution of the linear homing equations (14) predicts infinite missile acceleration near to intercept $(t \to t_f)$. It means that saturation is always reached and miss distances will be greater than predicted by linear analysis.

To evaluate optimal evasive tactics the mathematical formulation of the problem has to be modified. First, a new variable, the required missile lateral acceleration perpendicular to the line of sight $(\ddot{y}_M)_R$, is introduced. Using this variable, Eq. (9) is replaced by

$$(\ddot{y}_M)_R = F(s) \cdot (\ddot{y}_M)_c \tag{22}$$

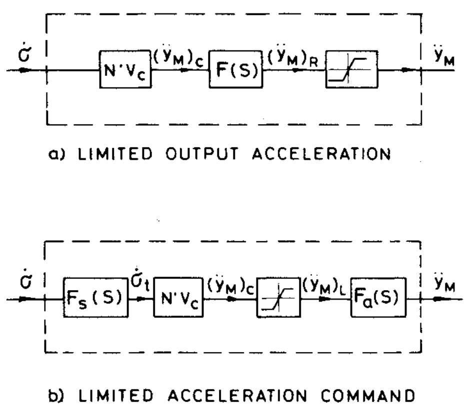

Fig. 2 Two types of missile acceleration limit.

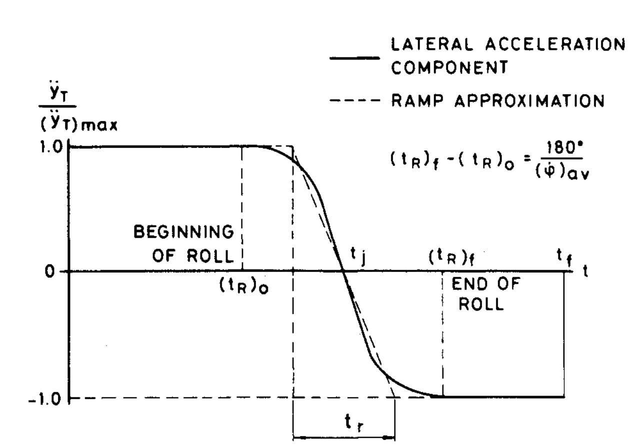

Fig. 3 Roll-rate-limited target maneuver model.

The nonlinear relationship between the actual and required missile maneuver is defined by

$$\ddot{y}_M = (\ddot{y}_M)_{\max} \operatorname{sat}\left\{ \frac{(\ddot{y}_M)_R}{(\ddot{y}_M)_{\max}} \right\} \tag{23}$$

Equations (22) and (23) implicitly assume that the limit on missile maneuverability is of the aerodynamic type, due to mechanical limits of control fin deflection or to hinge moment saturation. This limit is at the output of the guidance channel as shown in Fig. 2a. There are many missiles in which the limit is imposed on the acceleration command. This limit is in general sufficiently conservative, that an aerodynamic saturation, mentioned earlier, is not reached.

To model this type of guidance channel, the dynamics have to be separated into the "tracking loop" and the so-called "steering loop" as shown in Fig. 2b. Unless one of these loops is very fast compared to the other, so that its dynamics can be completely neglected, the assumption of first-order dynamics as used in Refs. 3, 4, and 6 is not adequate.

For a saturable guidance loop the state vector differential equation (14) is replaced by the nonlinear vector form

$$\dot{x} = f(x,t) + bu(t) \tag{24}$$

The payoff to be maximized (the square of the miss distance) is unchanged and the definition of the Hamiltonian, as in Eq. (18) remains valid. The costate vector λ, however, has to be determined in this case not by Eq. (19), which holds only for a linear system, but for the more general expression of

$$\dot{\lambda}_i = -(\partial H / \partial x_i) \quad (i = 1, 2, \ldots, n) \tag{25}$$

As a consequence of the validity of Eq. (18), exhibiting the linear dependence of the Hamiltonian on the control variable, Eq. (21), which predicts an optimal control function of the "bang-bang" type, remains also valid. However, in the nonlinear case, the costate vector and, consequently, the optimal control function are no longer independent of the trajectory. They should be computed by solving the two-point boundary-value problem defined by Eq. (24) and (25) with the boundary conditions stated in Eq. (20). Fortunately, because of the "bang-bang" structure of $u^*(t)$, a simple search technique can be used as an efficient alternative approach.

D. Target Dynamics

The "bang-bang"-type optimal control solution implicitly assumes that the direction of aircraft lateral acceleration can be changed instantaneously. Since aircrafts are not designed for high negative load factors, the direction reversal of the acceleration vector is obtained by a roll maneuver. The time to roll 180° has an order of magnitude of 1-3 sec. For missiles of slow dynamics this effect may be insignificant, but for high-performance guidance systems these few seconds are of the order of several time constants.

Simple analysis indicates that during a half-roll of the aircraft, the lateral acceleration in the plane of interest varies approximately like a ramp function as shown in Fig. 3.

The ramp time t_r is related to the average aircraft roll rate by

$$t_r = \alpha[\pi / (\dot{\phi})_{av}] \tag{26}$$

α being an adjustment coefficient with the value of the order of $0.5 < \alpha < 0.7$ depending on the roll-rate damping. This approximate formulation is used in the paper to model target dynamics effects.

Analysis and Results

Results are analyzed in the following order: 1) analysis of the optimal evasion for linear missile dynamics, including the effects of the dynamic structure of the guidance system; 2)

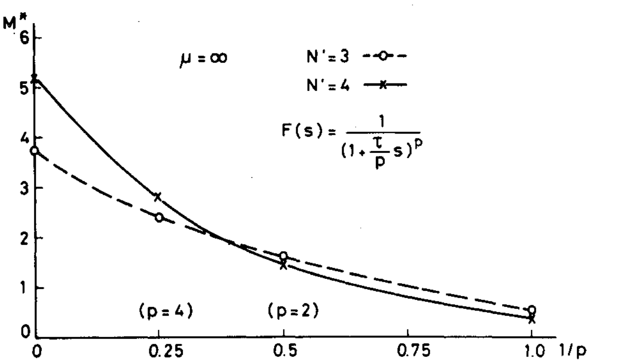

Fig. 4 Effect of system order on the optimal miss distance for linear systems.

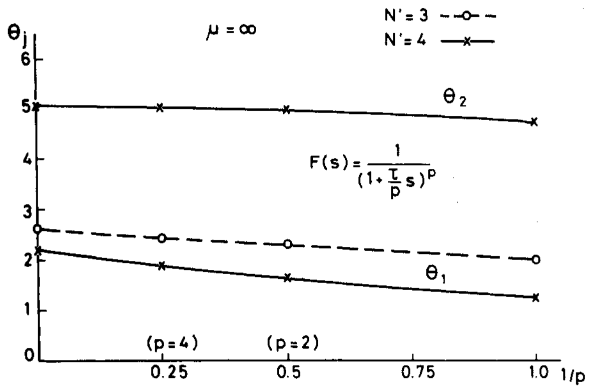

Fig. 5 Effect of system order on the timing of the optimal evasive maneuver for linear systems.

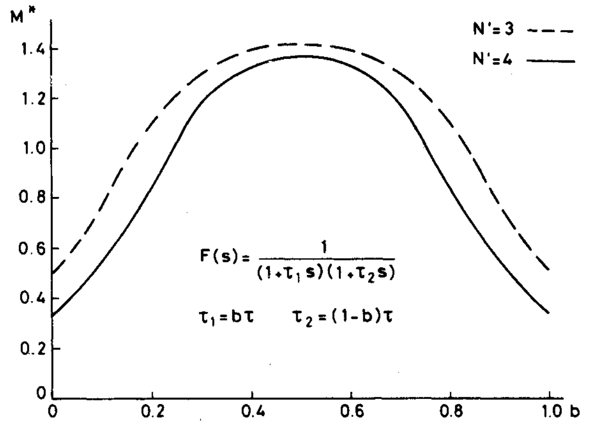

Fig. 6 Effect of the lag distribution on the optimal miss distance for a second-order overdamped linear system.

optimal evasion from missiles with limited maneuverability: effects of the missile-target maneuver saturation ratio and the location of the saturating element; and 3) effects of limited target roll-rate on the optimal evasion.

A. Linear Missile Dynamics

For linear dynamics of the first order, the costate vector differential equation (19) is solved in a closed form in the Appendix as a function of the normalized time to go

$$\theta \triangleq (t_f - t)/\tau = t_g/\tau \tag{27}$$

Those familar with the adjoint method of analysis[10] will recognize immediately that λ_l is the miss distance sensitivity function due to a target acceleration impulse, well-known in linear guidance theory[11-13]. (It is known to be equivalent to the miss distance sensitivity function due to a target velocity disturbance step).

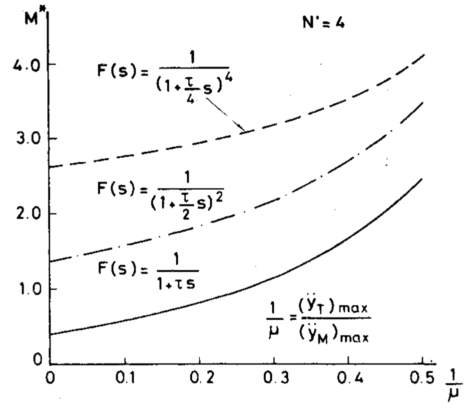

Fig. 7 Effect of limited missile maneuverability on the optimal miss distance.

For large values of θ, $\lambda_l(\theta)$ behaves as $\exp\{-\theta\}$. It changes sign $(N'-2)$ times for integer values of N'. (If N' is not integer the number of zeros, not including $\theta=0$, is $\mathrm{int}\{N'-1\}$). The order of system dynamics does not change this qualitative behavior. For any stable linear guidance system the optimal control is a bang-bang-type, nonsingular one.

Based on the observation made about the equivalence of λ_l and the miss distance sensitivity function of the adjoint analysis, there is no need to obtain the optimal miss distance by solving the trajectory equations (14) with the optimal control function (21). The normalized miss distance can be calculated directly from the miss distance sensitivity function due to a target acceleration step[12,13] which is the integral of $\lambda_l(\theta)$ with respect to θ,

$$\frac{m_0(\theta)}{\tau^2 \int \ddot{y}_T} = \int_0^\theta \lambda_l(\theta')\mathrm{d}\theta' = g(\theta) \tag{28}$$

Equation (21) indicates that whenever λ_l changes its sign, target acceleration is switched from $-(\ddot{y}_T)_{\max}$ to $+(\ddot{y}_T)_{\max}$ or vice versa (i.e., each time by a step of $|2(\ddot{y}_T)_{\max}|$. The optimal miss distance is, because of the linearity of the system, the sum of the miss distances generated by each step. The normalized optimal miss distance is given by

$$M^*(\theta) \triangleq \frac{m_0{}^*(\theta)}{\tau^2(\ddot{y}_T)_{\max}} = g(\theta) + 2\sum_j g(\theta_j) \tag{29}$$

where $\theta_j < \theta$ are defined by $\lambda_l(\theta_j) = 0$.

For $\theta > 10$ the first term can be neglected because of the strong damping of $\exp\{-\theta\}$, yielding

$$M^* \triangleq \frac{m_0{}^*}{\tau^2(\ddot{y}_T)_{\max}} \cong 2\sum_j g(\theta_j) \tag{30}$$

which is independent of the initial range represented by θ.

It was observed[7] and mentioned earlier that the qualitative behavior of $\lambda_l(\theta)$ is very similar for all stable transfer functions $F(s)$. This might have been the reason for the frequent use of first-order dynamic models. The approximation might be justified for preliminary guidance analysis, but it is by no means adequate for optimal evasion assessment. Both the optimal timing sequence and the resulting miss distance depend on the exact structure of the guidance transfer function $F(s)$.

The effects of the dynamic structure are demonstrated by two examples. First, let us consider the family of transfer functions of the form

$$F_p(s) = \frac{1}{[1+(\tau/p)s]^p} \quad (p=1,2,\ldots\infty) \tag{31}$$

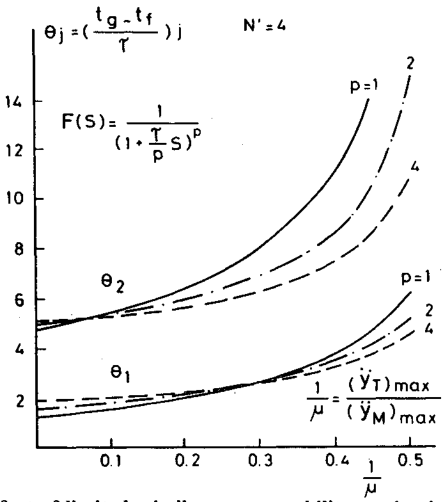

Fig. 8 Effect of limited missile maneuverability on the timing of the optimal evasive maneuver.

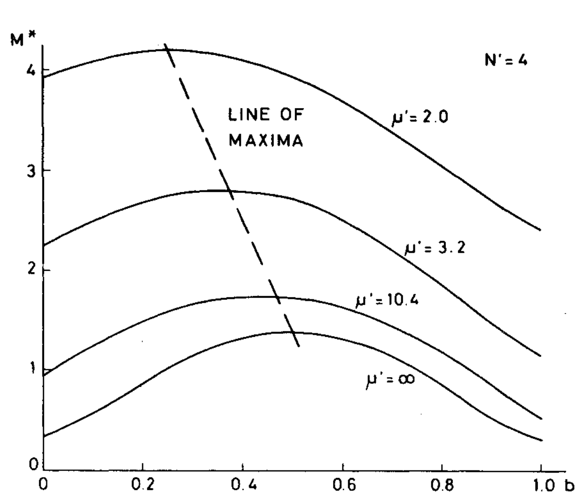

Fig. 9 Effect of lag distribution relative to the saturating element in a second-order system.

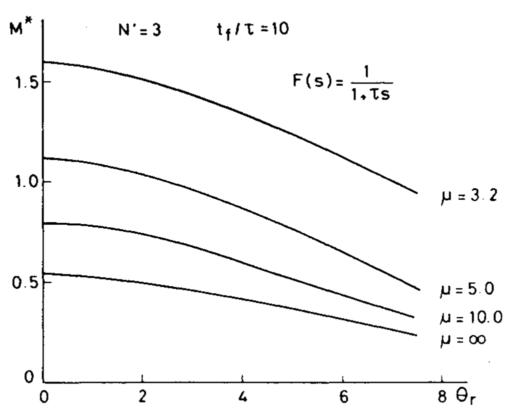

Fig. 10 Effect of roll-rate limit on the optimal miss distance.

having the same equivalent first-order time constant τ. The value of $p=\infty$ represents a pure delay

$$F_\infty(s) = \exp\{-\tau s\} \tag{32}$$

Results based on the solution of the adjoint system are presented for $N'=3$ and 4 as a function of $0<1/p<1$ in Figs. 4 and 5. They show slight variations on θ_j but a difference of an order of magnitude in the miss distances.

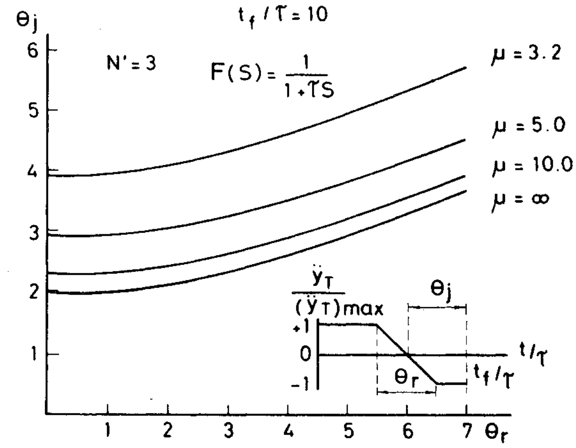

Fig. 11 Effect of roll-rate limit on the timing of the optimal evasive maneuver.

The second example is the family of second-order overdamped ($\zeta \geq 1$) systems with a transfer function of the form

$$F_b(s) = \frac{1}{(1+\tau_1 s)(1+\tau_2 s)} \tag{33}$$

with

$$\tau_1 = b\tau \quad \tau_2 = (1-b)\tau \quad 0 \leq b \leq 1$$

The normalized optimal miss distance is plotted as a function of b for $N'=3$, 4 in Fig. 6, indicating that a symmetric lag distribution ($b=0.5$) has the highest miss-distance sensitivity.

B. Saturated Missile Acceleration

Even for a missile with ideal dynamics ($F(s)\equiv 1$), a nonzero miss distance is generated if the missile target maneuver ratio μ defined by

$$\mu \triangleq \frac{(\ddot{y}_M)_{\max}}{(\ddot{y}_T)_{\max}} = \frac{V_{MR}(\dot{\gamma}_M)_{\max}}{V_{TR}(\dot{\gamma}_T)_{\max}} \tag{34}$$

does not satisfy the inequality

$$\mu \geq N'/(N'-2) \triangleq \mu_0(N') \tag{35}$$

The miss distance for constant target maneuver perpendicular to the line of sight is given in this case[14] by

$$(m_0)_s = \frac{\ddot{y}_T t_f^2}{2}\left(1-\frac{t_s}{t_f}\right)^{N'} \tag{36}$$

t_s being the time when missile saturation begins. The required lateral acceleration of a missile, guided by proportional navigation, against a constantly maneuvering target is

$$\frac{(\ddot{y}_M)_R}{\ddot{y}_T} = \frac{N'}{N'-2}\left[1-\left(1-\frac{t}{t_f}\right)^{N'-2}\right] \tag{37}$$

from which t_s is determined as

$$\frac{t_s}{t_f} = 1-\left(1-\frac{N'-2}{N'}\mu\right)^{1/(N'-2)} \tag{38}$$

Substituting Eq. (38) into Eq. (36) and using the definition of $\mu_0(N')$ yields, for the normalized miss distance,

$$\frac{(m_0)_s}{\ddot{y}_T t_f^2} = \frac{1}{2}\left(1-\frac{\mu}{\mu_0(N')}\right)^{\mu_0(N')} \tag{39}$$

For real system dynamics $[F(s)\neq 1]$ there is a very high sensitivity to the nonlinear phenomenon of maneuverability saturation. To demonstrate the effect of the saturation on optimal evasion, the case of an aerodynamically limited missile was solved by a search technique assuming "bang-bang"-type solutions.

Results obtained by this method for guidance transfer functions of the form (28) and $N'=4$, are presented in Figs. 7 and 8, as a function of the inverse maneuver ratio $1/\mu$. This presentation permits a direct comparison to the nonsaturated case ($\mu=\infty$).

Results can be summarized by the following points:

1) Limited missile lateral acceleration has a major effect both on the timing of the optimal evasion sequence and the resulting miss distance.

2) The lower the value of μ, the longer are the optimal durations of constant target maneuver periods.

3) The additional optimal miss distance component due to saturation

$$(m_0^*)_s = m_0^*(\mu) - m_0^*(\infty) \tag{40}$$

is more important for dynamics of lower order.

As a next step the case of saturated guidance command was considered. For a first-order system it has been demonstrated[3] that the optimal miss distance is about twice as great if the saturating element is ahead of the filter (the case of an ideal seeker), than if the filter preceeds the limiting component (i.e., ideal autopilot). For realistic modeling (see Fig. 2b) at least second-order dynamics has to be considered in a form similar to Eq. (33). The first-order approximation for the tracking loop is

$$F_t(s) = 1/(1+\tau_1 s) \tag{41}$$

and for the autopilot in closed loop we may assume

$$F_a(s) = 1/(1+\tau_2 s) \tag{42}$$

with

$$\tau_1 = b\tau \quad \tau_2 = (1-b)\tau \quad 0<b<1$$

In Fig. 9 the normalized optimal miss distances are plotted for $N'=4$ as a function of b for fixed values of μ' defined by

$$\mu' \triangleq (\ddot{y}_{Mc})_{\max}/(\ddot{y}_T)_{\max} \tag{43}$$

The line of $\mu'=\infty$ is replotted from Fig. 6 for comparison. It can be observed that the presence of a limit on the acceleration command has a different effect on the miss distance for different lag distributions. The line of maxima is shifted from $b=0.5$ for $\mu'=\infty$ to lower values as μ' becomes smaller. The values of $b=0$ and $b=1$ represent first-order systems with commanded and output acceleration limits, respectively.

C. Target Dynamics

Target dynamics is considered in this paper by taking into account the maximum roll-rate limit of the aircraft, resulting in a ramp approximation of the "bang-bang" maneuver.

Introducing the normalized ramp time

$$\theta_r = t_r/\tau \tag{44}$$

for t_r defined by Eq. (26), an example of a first-order missile with $N'=3$ and limited acceleration output was chosen. Results are presented on Figs. 10 and 11 as a function of θ_r for given values of μ.

It can be concluded that limited aircraft roll rate has a substantial effect, reducing the maximum attainable miss distances. The normalized time to go to initiate the rolling maneuver is given by

$$(\theta_R)_0 = \theta_j(\theta_r,\mu) + \theta_r/2\alpha \tag{45}$$

Note that the values of θ_j are shifted from the optimal value determined for an infinite roll rate ($\theta_r=0$).

Concluding Remarks

This paper presents a simple and efficient method to determine optimal evasive maneuvers from homing missiles guided by proportional navigation. Following the principle ideas for linear guidance analysis, the optimal evasive maneuver sequence is determined as a function of the normalized time to go "θ" only. The resulting normalized miss distance M^*, and the timing of the optimal manuevers depend on several factors: 1) the exact dynamic structure of the guidance system (system order and lag distribution), 2) the missile-target maneuver ratio μ, 3) the location of the saturating element in the guidance loop, and 4) target dynamics (represented by limited roll rate).

These factors, neglected in most analytical studies, determine the order of magnitude of the miss distance and therefore the effectiveness of the evasive maneuver. It is strongly recommended to include them in any future study endeavoring for meaningful results.

This work was carried out as a two-dimensional analysis, assuming linearized kinematics. To check the validity of the linearization a comparison was made with a two-dimensional nonlinear kinematic model.[6] The results of the nonlinear kinematic model depend on the initial conditions γ_{T_0} and R_0. However, the terminal phase of the optimal evasive maneuver is almost always of the "bang-bang" type as predicted by Eq. (21).

For small values of γ_{T_0} satisfying $\cos(\gamma_{T_0})\approx 1$ the results are identical. For larger initial values of γ_{T_0}, the optimal maneuver of the nonlinear model has the tendency to reduce the absolute value of γ_T (to increase $\cos\gamma_T$). If the initial conditions permit reducing γ_T sufficiently, the resulting miss distance is similar to the one obtained in the linearized analysis. If not, the miss distance is rather small, which indicates either that this "optimal" maneuver is inefficient, or that the two-dimensional analysis is misleading. Indeed, in these cases the optimal evasive maneuver has to be performed *out of the plane* of interest, requiring a three-dimensional analysis.

The comparison gave a complete a posteriori justification to the basic assumptions (nos. 3-5) of this paper, demonstrating that for the assessment of an optimal evasion, the domains of validity of linearized kinematics and two-dimensional analysis are identical. Both assumptions limit the usefulness of the results to engagements with nearly "head-on" or "tail-chase" initial conditions ($\cos\gamma_{T_0}\approx 1$).

For more general initial conditions a three-dimensional model has to be used. There is good evidence that the method presented in this paper can be extended without difficulty for linearized three-dimensional analysis. Because of its simplicity, this method is an attractive candidate for real-time onboard computations.

Appendix: Closed-Form Solution of the Costate Vector for Linear First-Order Dynamics

The state equations of a linearized homing process are written in vector form as

$$\dot{x}(t) = A(t)x(t) + bu(t) \tag{A1}$$

For a guidance transfer function of the first order

$$\frac{\dot{\gamma}_M}{(\dot{\gamma}_M)_c} = F(s) = \frac{1}{1+\tau s} \tag{A2}$$

the state vector has four components

$$x(t) = \mathrm{col}[x_1,(t),x_2(t),x_3(t),x_4(t)]$$
$$\triangleq \mathrm{col}[\Delta\gamma_T(t),y(t),\Delta\gamma_M(t),\dot{\gamma}_M(t)] \tag{A3}$$

The set of differential equations for this system consists of Eqs. (11), (5), and the combination of (7), (8), and (A2), defining the state matrix

$$A(t)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ V_{TR} & 0 & -V_{MR} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{N' V_{TR}}{\tau V_{MR}(t_f-t)} & \dfrac{N'}{\tau V_{MR}(t_f-t)^2} & \dfrac{-N'}{\tau(t_f-t)} & \dfrac{-1}{\tau} \end{bmatrix} \tag{A4}$$

and the vector $\boldsymbol{b}$

$$\boldsymbol{b}=(\dot{\gamma}_T)_{\max}\mathrm{col}[1,0,0,0] \tag{A5}$$

For the linear optimal control problem, described in the first part of this paper, the costate vector

$$\lambda(t)=\mathrm{col}[\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t)] \tag{A6}$$

is determined by the adjoint equation

$$\dot{\lambda}(t)=-A^T(t)\lambda(t) \tag{A7}$$

and the terminal boundary conditions

$$\lambda(t_f)=\mathrm{col}[0,2m_0,0,0] \tag{A8}$$

Introducing the normalized "time to go" as a new independent variable

$$\theta \triangleq (t_f-t)/\tau=t_g/\tau \tag{A9}$$

Eq. (A7) becomes

$$\frac{\mathrm{d}\lambda(\theta)}{\mathrm{d}\theta}=A^T(\theta)\lambda(\theta) \tag{A10}$$

and the terminal conditions of Eq. (A8) are transformed to initial conditions

$$\lambda(\theta=0)=\mathrm{col}[0,2m_0,0,0] \tag{A11}$$

Equation (A10) with (A11) can be solved independently of the missile trajectory equation (A1). The vector differential equation (A10) is written in scalar form as

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\theta}=(\tau V_{TR})\lambda_2+\left(\frac{N' V_{TR}}{\tau V_{MR}\theta}\right)\lambda_4 \tag{A12}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}\theta}=\left(\frac{N'}{\tau^2 V_{MR}\theta^2}\right)\lambda_4 \tag{A13}$$

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}\theta}=-(\tau V_{MR})\lambda_2-\left(\frac{N'}{\tau\theta}\right)\lambda_4 \tag{A14}$$

$$\frac{\mathrm{d}\lambda_4}{\mathrm{d}\theta}=\tau\lambda_3+\lambda_4 \tag{A15}$$

Comparing Eq. (A14) to Eq. (A12) indicates that

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}\theta}=-\frac{V_{MR}}{V_{TR}}\frac{\mathrm{d}\lambda_1}{\mathrm{d}\theta} \tag{A16}$$

Differentiating Eq. (A15) and substituting Eq. (A16) yields

$$\frac{\mathrm{d}^2\lambda_4}{\mathrm{d}\theta^2}+\frac{\mathrm{d}\lambda_4}{\mathrm{d}\theta}=-\frac{\tau V_{MR}}{V_{TR}}\frac{\mathrm{d}\lambda_1}{\mathrm{d}\theta} \tag{A17}$$

Substituting Eq. (A12) and differentiating again we obtain

$$\frac{\mathrm{d}^3\lambda_4}{\mathrm{d}\theta^3}+\frac{\mathrm{d}^2\lambda_4}{\mathrm{d}\theta^2}+\tau^2 V_{MR}\frac{\mathrm{d}\lambda_2}{\mathrm{d}\theta}+\frac{N'}{\theta}\left(\frac{\mathrm{d}\lambda_4}{\mathrm{d}\theta}-\frac{\lambda_4}{\theta}\right)=0 \tag{A18}$$

Using Eq. (A13) cancels the third and the last terms and yields a second-order linear differential equation for $\mathrm{d}\lambda_4/\mathrm{d}\theta$.

$$\theta\frac{\mathrm{d}^3\lambda_4}{\mathrm{d}\theta^3}+\theta\frac{\mathrm{d}^2\lambda_4}{\mathrm{d}\theta^2}+N'\frac{\mathrm{d}\lambda_4}{\mathrm{d}\theta}=0 \tag{A19}$$

This equation can be solved in a closed form using Laplace transforms. By introducing

$$h(s)=s\lambda_4(s)=\mathcal{L}\left\{\frac{\mathrm{d}\lambda_4}{\mathrm{d}\theta}\right\} \tag{A20}$$

the Laplace transform of Eq. (A19) leads to a first-order homogeneous differential equation

$$\frac{\mathrm{d}h(s)}{\mathrm{d}s}=\frac{(N'-1)-2s}{s(s+1)}h(s) \tag{A21}$$

with the immediate solution

$$h(s)=s\lambda_4(s)=C\frac{s^{N'-1}}{(s+1)^{N'+1}} \tag{A22}$$

Table 1 Components of the costate vector for a linear first-order system[a]

N'	$f_1(\theta)$	$f_2(\theta)$	$f_4(\theta)$
3	$\theta-\theta^2/2$	$1-\theta/2$	$\theta^2-\theta^3/6$
4	$\theta-\theta^2+\theta^3/6$	$1-\theta+\theta^2/6$	$\theta^2/2-\theta^3/3+\theta^4/24$
5	$\theta-(3/2)\theta^2+\theta^3/2-\theta^4/24$	$1-(3/2)\theta+\theta^2/2-\theta^3/24$	$\theta^2/2-\theta^3/2+\theta^4/8-\theta^5/120$
6	$\theta-2\theta^2+\theta^3-\theta^4/6+\theta^5/120$	$1-2\theta-4\theta^2-\theta^3/6+\theta^4/120$	$\theta^2/2-(3/2)\theta^3+\theta^4/4-\theta^5/30+\theta^6/720$

[a] $\lambda_1(\theta)=2m_0\tau V_{TR}\exp\{-\theta\}f_1(\theta)$; $\lambda_2(\theta)=2m_0\exp\{-\theta\}f_2(\theta)$; $\lambda_3(\theta)=-(V_{MR}/V_{TR})\lambda_1(\theta)$; $\lambda_4(\theta)=-2m_0\tau^2 V_{MR}\exp\{-\theta\}f_4(\theta)$.

Table 2 Optimal switching time, normalized miss-distance sensitivity, and optimal normalized miss distance for a first-order linear system[a]

N'	θ_j	$g_{N'}(\theta)$	M^*
3	2.0	$\theta^2/2$	0.541
4	1.47, 4.73	$\theta^2/2-\theta^3/6$	0.375
5	0.94, 3.30, 7.76	$\theta^2/2-\theta^3/3+\theta^4/24$	0.292
6	0.74, 2.57, 5.74, 10.94	$\theta^2/2-\theta^3/3+\theta^4/8-\theta^5/120$	0.244

[a] $g(\theta)=\int_0^\theta \lambda_1(\theta')\mathrm{d}\theta'=2m_0\tau V_{TR}\exp\{-\theta\}g_{N'}(\theta)$; $M^*=\dfrac{m_0^*}{\tau^2(\ddot{y}_T)_{\max}}=2\sum_j g(\theta_j)$; $\lambda_1(\theta_j)=0$, $(j=1,\ldots N'-2)$.

where C is the constant of integration to be determined later.

The Laplace transform of Eq. (A17) yields for $\lambda_1(s)$

$$\lambda_1(s) = -C\frac{V_{TR}}{\tau V_{MR}}\cdot\frac{s^{N'-2}}{(s+1)^{N'}} = C_1\frac{s^{N'-2}}{(s+1)^{N'}} \qquad \text{(A23)}$$

Since the rules of inverse Laplace transform indicate that

$$\mathcal{L}^{-1}\left\{\frac{s^r}{(s+1)^q}\right\} = \frac{d^r}{d\theta^r}\left\{e^{-\theta}\frac{\theta^{q-1}}{(q-1)!}\right\} \qquad \text{(A24)}$$

$\lambda_1(\theta)$ is given for integer values of N' in a form

$$\lambda_1(\theta) = C_1 e^{-\theta}[f_1(\theta)] \qquad \text{(A25)}$$

where $f_1(\theta)$ is a polynomial. The remaining components $\lambda_2(\theta)$, $\lambda_3(\theta)$, $\lambda_4(\theta)$ have the same structure and can be determined by Eqs. (A12), (A16), and (A22), respectively.

The initial conditions (A11) are satisfied automatically by this solution, determining the constant C_1 as

$$C_1 = \tau V_{TR}\lambda_2(\theta) = 2\tau V_{TR} m_0 \qquad \text{(A26)}$$

The components of the costate vector $\lambda(\theta)$ for this first-order example are summarized in Table 1 for several values of N'.

The function $\lambda_1(\theta)$ plays the role of the switch function [see Eq. (21)] which is equivalent to the normalized miss-distance sensitivity function for target acceleration impulse. The normalized miss-distance sensitivity function due to a target maneuver step is therefore

$$g(\theta) = \int_0^\theta \lambda_1(\theta')\,d\theta' \qquad \text{(A27)}$$

As indicated by Eq. (21), target acceleration direction has to be switched at the values of θ_j defined by

$$\lambda_1(\theta_j) = 0 \qquad (j = 1,\ldots N'-2) \qquad \text{(A28)}$$

and in this case the optimal miss distance is given by Eq. (30). The function $g(\theta)/\exp\{-\theta\}$, the corresponding values of θ_j and the optimal normalized miss distance are given in Table 2 for the example of a first-order system.

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